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LETTER TO THE EDITOR

Crossover in the one-dimensional self-directed walk

Loïc Turban and Jean-Marc Debierre

Laboratoire de Physique du Solide, Université de Nancy I, BP 239, F54506 Vandoeuvre les Nancy, France†

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Abstract. The self-directed walk is studied in one dimension. In this walk with memory the jump probability is given by $W_{N\pm}(i)=[1+\exp(\pm g\Delta_N(i))]^{-1}$ where Δ_N is the difference between the number of times the sites in the forward and backward directions have been visited after N steps. When $g>0$ there is a crossover between a Gaussian random walk and an intermediate regime where the radius of gyration grows like N^2 followed by a crossover to the asymptotic regime where the walk is directed. When $g<0$ a single crossover is obtained between the Gaussian random walk and a saturation regime at large N when the walk is self-attracting.

The study of random walks with memory has been a field of great activity in recent years (see Lyklema (1986) and Peliti and Pietronero (1987) for reviews). This renewed interest originates in the work on the true self-avoiding walk (TSAW) which is a dynamical version of the old self-avoiding walk (SAW) problem (Amit *et al* 1983, Pietronero 1983, Obukhov and Peliti 1983, Obukhov 1984, Rammal *et al* 1984, Bernasconi and Pietronero 1984, Stella *et al* 1984, Öttinger 1985).

Recently, we introduced and studied a new kind of growing random walk, the self-directed walk (SDW) in which the walker is allowed to jump with the same probability in any lattice direction where a path is open (Turban and Debierre 1987a, b). An open path is a lattice direction in which no site has been previously visited by the walker. A self-consistent Flory-like argument has been developed (Turban and Debierre 1987a) leading to the following values of the radius of gyration exponent:

$$\begin{aligned} \nu &= 1 && 1\leqslant d\leqslant 2 \\ \nu &= 2/d && 2\leqslant d\leqslant 4 \\ \nu &= \tfrac{1}{2} && d\geqslant d_c=4. \end{aligned} \tag{1}$$

Our Monte Carlo results in two and three dimensions are in good agreement with the Flory predictions. Compared to the SAW ($\nu=3/(d+2)$; $d_c=4$) and the TSAW ($\nu=2/(d+2)$; $d_c=2$) the SDW is more elongated below the upper critical dimension $d_c=4$ at which it becomes Gaussian. It is directed, in the sense that the radius of gyration R_N grows like N at and below $d=2$.

† Unité Associée au CNRS no 155.

In one dimension with the rules given above the problem is trivial. In the present work they are modified in analogy with the TSAW (Amit *et al* 1983). The walker at $\boldsymbol{r}$ is allowed to jump in the forward (+) or backward (−) lattice direction (α) with a probability

$$W_{\pm}^{(\alpha)}(\boldsymbol{r}) = \exp(-gN_{\pm}^{(\alpha)}(\boldsymbol{r}))\left(\sum_{\alpha=1}^{2d}[\exp(-gN_{+}^{(\alpha)}(\boldsymbol{r}))+\exp(-gN_{-}^{(\alpha)}(\boldsymbol{r}))]\right)^{-1} \tag{2}$$

where $N_{\pm}^{(\alpha)}(\boldsymbol{r})$ gives the number of times the sites in the corresponding direction have already been visited. The old rules are recovered in the limit $g \to +\infty$. Although the asymptotic behaviour is not changed, in one dimension when $g > 0$, one gets a succession of two crossovers before the asymptotic directed regime is entered. Moreover this formulation allows the study of the regime $g < 0$ where the walk is self-attracting.

In one dimension equation (2) may be written

$$\begin{aligned} W_{N\pm}(i) &= [1+\exp(\pm g\Delta_N(i))]^{-1} \\ \Delta_N(i) &= \sum_{j>i} n_N(j) - \sum_{j<i} n_N(j) \end{aligned} \tag{3}$$

where $n_N(j)$ gives the occupation number of site j, i.e. the number of times site j has been visited by the walker after N steps. The walks are grown on a linear chain using a standard Monte Carlo method (Binder 1979). At each Monte Carlo step N, the end-to-end square radius $(x_N - x_0)^2$ and the number of distinct sites visited are stored. Averages $X_N^2 = \langle (x_N - x_0)^2 \rangle$ and S_N are taken over 10^4 walks of 10^4 steps for the 10 values of g studied ($0.0001 < g < 1$ and $-0.01 < g < 0.0001$). The mean site occupation number $\langle n_N(x) \rangle$ after 1000 steps is shown in figure 1 for three values of g. When $g = -0.001$ one gets a sharp central peak as a result of the self-attraction whereas when $g = 0.001$ a tail begins to develop since the walk is directed at large N.

The averaged end-to-end radius X_N is shown in figure 2 as a function of N with logarithmic scales. When $g > 0$ one gets first a Gaussian random walk (GRW) regime $X_N \sim N^{1/2}$ for small N and g, then a crossover to an intermediate regime where X_N grows quickly with N like N^2 and finally a second crossover to the asymptotic regime

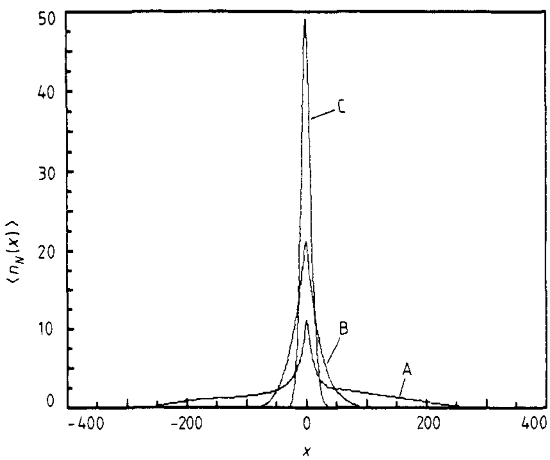

Figure 1. Mean occupation number $\langle n_N(x) \rangle$ after $N = 1000$ steps for three values of g: (A) 0.001; (B) 0.0001; (C) −0.001.

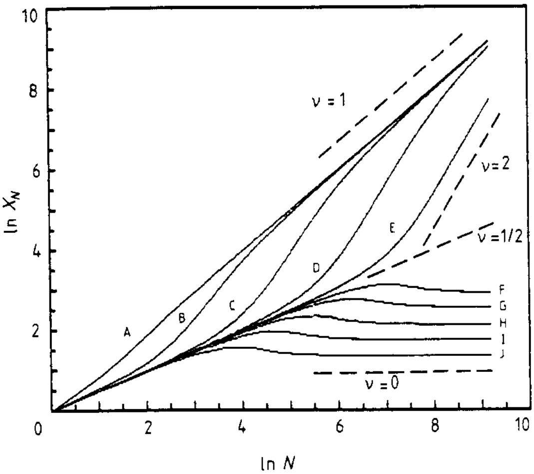

Figure 2. Variation of ln X_N with ln N for various values of g: (A) 1; (B) 0.1; (C) 0.01; (D) 0.001; (E) 0.0001; (F) −0.0001; (G) −0.0003; (H) −0.001; (I) −0.003; (J) −0.01. The slopes give the radius of gyration exponent ν in the different regimes.

where the walk is directed $X_N \sim N$. When $g<0$ the walk is self-trapping at large N and one gets a single crossover between the GRW regime at small N and g and a saturation regime with $X_\infty(g) \sim (-g)^{-\nu'}$ when $N \to \infty$. $X_\infty(g)$ is obtained through an extrapolation of X_N^2 against N^{-1} to $N^{-1}=0$. A least-squares fit gives $2\nu' = 0.670 \pm 0.005$. The same behaviour is obtained with S_N as expected in one dimension where S_N scales like X_N (figure 3). When $g<0$, $S_\infty(g) \sim (-g)^{-s'}$ with $s' = 0.332 \pm 0.002$.

These results may be understood using a master equation approach, dimensional analysis and scaling arguments as in the TSAW (Obukhov 1984, Rammal *et al* 1984, Bernasconi and Pietronero 1984). Let $P_N(i)$ be the probability for the walker to visit site i after N steps. $P_N(i)$ satisfies the master equation

$$P_{N+1}(i) = P_N(i-1)\,W_{N+}(i-1) + P_N(i+1)\,W_{N-}(i+1). \qquad (4)$$

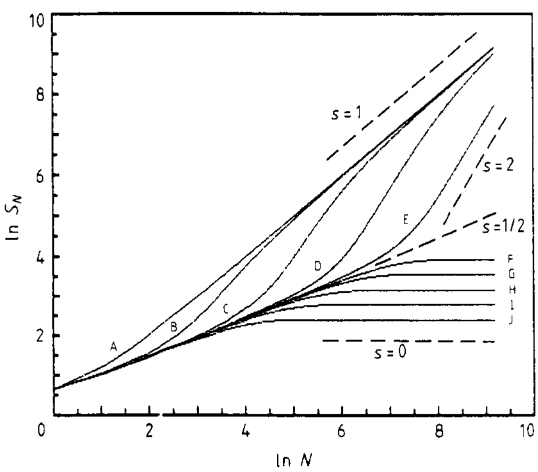

Figure 3. Variation of ln S_N with ln N for the g values indicated in figure 2. The slopes give the exponent s in $S_N \sim N^s$ in the different regimes.

In the weak-coupling regime ($g\Delta_N < 1$) equation (3) is

$$W_{N\pm}(i) \simeq \tfrac{1}{2}(1 \mp \tfrac{1}{2}g\Delta_N(i)) \tag{5}$$

so that in the continuum approximation, keeping the leading contributions, one gets

$$\frac{\partial P(x, N, g)}{\partial N} = \frac{1}{2}\frac{\partial^2 P(x, N, g)}{\partial x^2} + \tfrac{1}{2}g\frac{\partial}{\partial x}(\Delta(x, N, g)P(x, N, g)] \tag{6}$$

with

$$\Delta(x, N, g) = \int_x^\infty n(y, N, g)\,\mathrm{d}y - \int_{-\infty}^x n(y, N, g)\,\mathrm{d}y. \tag{7}$$

In the GRW regime $X^2 = \langle x^2 \rangle \sim N$, $P \sim N^{-1/2}$ and $\Delta \sim N$ (see the appendix) and equation (6) leads to

$$1/N \sim 1/X^2 + gN/X \sim (1/X^2)(1 + gN^{3/2}). \tag{8}$$

It follows that the appropriate small parameter in a perturbation expansion is $z = |g|N^{3/2}$. The scaling ansatz for the mean square radius is then (see the appendix)

$$X_N^2 = Nf(z) \tag{9}$$

with

$$\begin{aligned} f(z) &= a_0 + a_1 z + \cdots \qquad & z<1 \\ f(z) &\sim z^\omega & z>1. \end{aligned} \tag{10}$$

When $g < 0$ and $N \to \infty$ in the saturation regime

$$X_\infty^2 \sim N^0 \sim (-g)^\omega N^{3\omega/2+1} \tag{11}$$

so that $\nu' = -\frac{1}{2}\omega = \frac{1}{3}$ in agreement with the numerical results. The scaling function $f(z)$ is shown in figure 4. GRW behaviour is obtained when $|g|N^{3/2} < 1$. When $g<0$ the

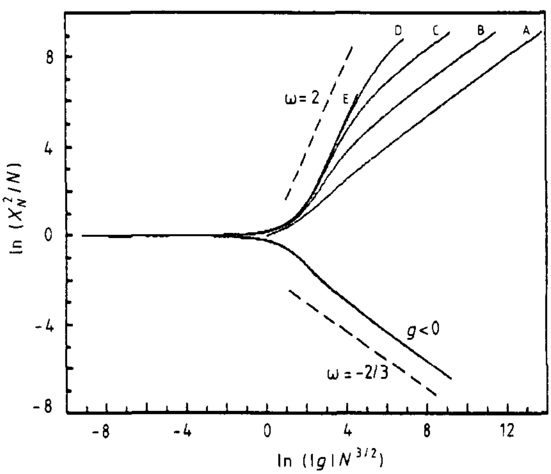

Figure 4. Scaling function $X_N^2/N = f(|g|N^{3/2})$ showing the crossover between GRW and intermediate regime when $g > 0$ and between GRW and saturation regime when $g < 0$. The slopes give the exponent ω defined in equation (10). The g values are the same as in figure 2.

crossover is towards the saturation regime ($f(z) \sim z^{-2/3}$) whereas when $g > 0$ one gets first an intermediate regime in the weak-coupling region ($gN \leq 1$) as long as equation (5) remains valid. There the diffusion term in equation (8) may be neglected and

$$X_N^2 \sim g^2 N^4 \tag{12}$$

so that $f(z) \sim z^2$. When $g > 0$ and $gN \geq 1$ the asymptotic SDW regime is reached and $X_N^2 \sim N^2$. The small parameter is then e^{-gN} and the variable which is appropriate to describe the new crossover is $t = gN$ with the scaling ansatz

$$\begin{aligned} &X_N^2 = N^2 h(t) \\ &h(t) \to 1 \qquad t \to \infty \\ &h(t) \sim t^2 \qquad t \leq 1 \end{aligned} \tag{13}$$

according to equation (12). The numerical results support this assumption as shown in figure 5. The same scaling behaviour is obtained with $f' = S_N^2/N$ and $h' = S_N^2/N^2$.

To conclude let us mention that we intend to complete the study of the SDW in two and three dimensions along the same lines.

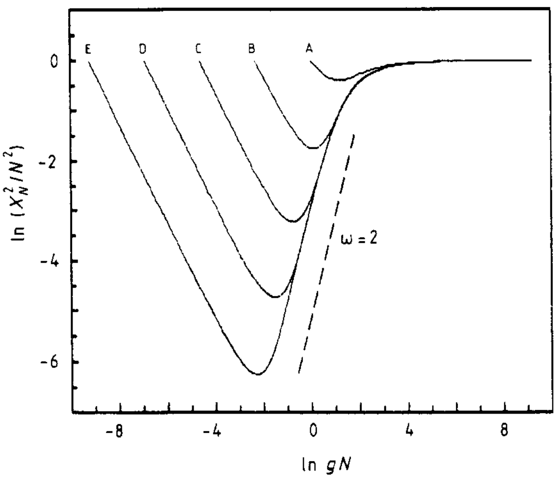

Figure 5. Scaling function $X_N^2/N^2 = h(gN)$ showing the crossover between the intermediate regime in the weak-coupling region and the SDW asymptotic regime for the positive g values given in figure 2.

Appendix

Let us assume that P is a generalised homogeneous function of x, N and g transforming under a change of the length scale by a factor b as

$$P' = P(x', N', g') = b^{y_P} P(x, N, g) \tag{A1}$$

where

$$x' = b^{-1} x \qquad N' = b^{y_N} N \qquad g' = b^{y_g} g. \tag{A2}$$

From the normalisation condition on P and P' one easily deduces

$$y_p = 1. \tag{A3}$$

Since $P = \partial n/\partial N$, $n(x, N, g)$ transforms as

$$n' = b^{1+y_N} n \tag{A4}$$

and since according to equation (7) $n = -\frac{1}{2}\partial\Delta/\partial x$, $\Delta(x, N, g)$ transforms as

$$\Delta' = b^{y_N}\Delta. \tag{A5}$$

When these results are used in equation (7), in the weak-coupling regime, one obtains

$$b^{y_N-1}\frac{\partial P'}{\partial N'} = b^{-3}\frac{\partial^2 P'}{\partial x'^2} + b^{-y_g-y_N-2} g' \frac{\partial(\Delta' P')}{\partial x'}. \tag{A6}$$

Since b is arbitrary, the exponents in front of the three terms must be the same so that

$$y_N = -2 \qquad y_g = 3. \tag{A7}$$

Equations (A1), (A4) and (A5) with $b = N^{1/2}$ give

$$P(x, N, g) = N^{-1/2}P(xN^{-1/2}, 1, gN^{3/2}) \tag{A8}$$

$$n(x, N, g) = N^{1/2}n(xN^{-1/2}, 1, gN^{3/2}) \tag{A9}$$

$$\Delta(x, N, g) = N\Delta(xN^{-1/2}, 1, gN^{3/2}) \tag{A10}$$

whereas using equation (A8)

$$\begin{aligned} X_N^2 &= \int_{-\infty}^{+\infty} \mathrm{d}x\, x^2 P(x, N, g) \\ &= N\int_{-\infty}^{+\infty} \mathrm{d}y\, y^2 P(y, 1, gN^{3/2}) \\ &= Nf(gN^{3/2}) \end{aligned} \tag{A11}$$

in agreement with equation (9).

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